Vectors

We will be using vectors and matrices to store and manipulate data.

Definition: A **vector** $\vec{\mathbf{v}}$ is a column of numbers. Use bold faced letters or vector signs to distinguish vectors from other 1-D variables.

We refer to the **entries** of a vector by using subscripts.

The **length** of a vector is the number of entries it has. (normally n)

Example. $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T.$

Example. Use a vector to represent the age distribution of a population: let F_i be the number of females with ages in the interval $\big[5i, 5(i+1)\big)$. We can represent the total female population by the vector $\vec{\mathbf{F}}$.
The females from 0 up to 5 are counted in F_0;
those from 5 up to 10 are counted in F_1, etc.

$$\vec{\mathbf{F}} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{bmatrix}$$

Matrices

Definition: A **matrix** A is a two-dimensional array of numbers.

A matrix with m rows and n columns is called an $\overset{\text{“}m\text{ by }n\text{”}}{m \times n}$ matrix.

$\star$ Row by column — Row by column — Row by column $\star$

Note: A vector can be thought of as an $n \times 1$ matrix.

Matrices are denoted by a capital letter. Entries are lower case and have two subscripts, the corresponding row and column.

Example. A generic 2×3 matrix has the form $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$.

Definition: The matrix $B = \begin{bmatrix} 30 & 50 \\ 100 & 250 \end{bmatrix}$ is a **square matrix** because it has the same number of rows as columns.

Matrices

Example. We will sometimes interpret a matrix as a **transition** matrix.

In this case, the matrix is square (say $n \times n$), where the n rows and n columns correspond to certain **states** (situations).

An entry $a_{i,j}$ represents transitioning <u>from state j to state i</u>.

Example. In our population example, suppose we want to model people getting older, transitioning from one state (age group) to the next. We would set up a transition matrix such as:

$$\begin{array}{cc} & \textbf{FROM}\text{ state:} \\ \textbf{TO}\text{ state:} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

because everyone in the first age group will move to the second age group ($a_{2,1}$), everyone in state 2 will move to state 3 ($a_{3,2}$), etc.

We will soon see why this is "backwards".

Matrix Multiplication

The power of matrices arises in their multiplication.

Given two matrices, A of size $m \times k$ and B of size $l \times n$, we can find the product AB **if and only if** k equals l.

Let A be an $m \times k$ matrix and B, $k \times n$. Then AB is of size $m \times n$.

To calculate the entries of AB, remember: "Row by column":

$$\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 6 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \end{bmatrix}$$

When we write A^2, this means AA; A^3 means AAA, etc.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \bigcirc & \bigcirc \\ 0 & 1 & \bigcirc \\ 0 & 0 & 1 \end{bmatrix}$$

The power of transition matrices

Example. Modeling a changing population using a matrix model.

Let us choose a size of age interval Δ=5 years ("Delta"), and divide the female population into states:

age distribution vector:

State 0: ages $[0,5)$ with $F_0 = 150$ females
State 1: ages $[5,10)$ with $F_1 = 200$ females
State 2: ages $[10,15)$ with $F_2 = 180$ females
State 3: ages $[15,20)$ with $F_3 = 120$ females
State 4: ages $[20,25)$ with $F_4 = 60$ females

$$\vec{\mathbf{F}} = \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix}$$

Using a transition matrix, we can determine the population in 5 years:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^1 \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \\ 200 \\ 180 \\ 120 \end{bmatrix}$$

The transition matrix is "backwards" because:

Leslie Matrices

The transition matrix in the previous example is not entirely realistic, because people die and are born

To take death into account, modify:

To take birth into account, modify:
(¡ females !)

The resulting transition matrix is called a **Leslie matrix**:
Let m_i be the average number of females that women in state i bear.
Let p_i be the fraction of women in state i that survive to state $i+1$.

$$\text{then} \begin{bmatrix} F_0(t+\Delta) \\ F_1(t+\Delta) \\ F_2(t+\Delta) \\ \vdots \\ F_{n-1}(t+\Delta) \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ p_0 & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & p_{n-2} & 0 \end{bmatrix} \begin{bmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_{n-1}(t) \end{bmatrix}$$

$$\vec{\mathbf{F}}(t+\Delta) = M\vec{\mathbf{F}}(t)$$

Leslie Matrices

Example. An animal population example (p. 47)
The population in three age groups, $F_0 = 80$, $F_1 = 40$, and $F_2 = 20$.

Suppose that as Δ time passes, everyone in state 2 dies, and one quarter of everyone else dies. Also suppose that the age-specific maternity rates are $m_0 = 0$, $m_1 = 1$, and $m_2 = 2$. Determine the Leslie matrix and the population distributions at times Δ and 2Δ.

$$\begin{bmatrix} & & \\ & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} = \vec{\mathbf{F}}(\Delta)$$

$$\begin{bmatrix} & & \\ & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} = \vec{\mathbf{F}}(2\Delta)$$

Leslie Matrices

Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(b) Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be any initial population. Find a and b so that

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(c) Find $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ using parts (a) and (b).

(d) Show that the total population $P_n \approx P_0 2^n$.

- A Leslie matrix model is more descriptively realistic than the exponential model from Section 1.4, yet gives the same results.
- We've just worked with eigenvalues and eigenvectors!